

On the Solution of Gauss Circle Problem Conjecture (Revised)

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Abstract

We give a proof of a mean value asymptotic formula for the number of representations of an integer as sum of two squares known as the Gauss circle problem.

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1 Introduction

In geometry the number of lattice points lying in the circle

$$x^2 + y^2 = n \quad (1)$$

is 'say' $r_2(n)$. The equivalent of the above proposition in number theory is saying, the number of representations of the integer n as a sum of two squares is $r_2(n)$. Hence evaluating the function $r_2(n)$ has both geometrical and number theoretic meaning. Gauss itself prove that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} r_2(n) = \pi. \quad (2)$$

The problem we are describing, firstly began by finding the best possible constant $\theta > 0$ of the asymptotic expansion

$$\sum_{n \leq x} r_2(n) = \pi x + O(x^{\theta+\epsilon}), \forall \epsilon > 0, x \rightarrow \infty \quad (3)$$

The first result it was that of Gauss which gave a first value $\theta = \frac{1}{2}$. For an analytic geometry proof of this case one can see [2].

After Gauss many other scientists try to give better estimates with the most recent that of Huxley $\theta = \frac{131}{416} \approx 0.3149$ (see [11]) in 2003. Gauss circle problem (conjecture) states that $\theta = \frac{1}{4}$ in (3).

There is a conversion procedure due to Richert that expresses the Gauss circle problem in terms of divisor problem of Dirichlet i.e

$$\Delta(x) = \sum_{n \leq x} d(n) - (\log x + 2\gamma - 1)x, \text{ where } d(n) = \sum_{d|n} 1 \quad (4)$$

The conjecture for the Dirichlet divisor problem is

$$\Delta(x) = O\left(x^{1/4+\epsilon}\right), \forall \epsilon > 0, x \rightarrow \infty \quad (5)$$

A result of Hardy shows that

$$\limsup_{x \rightarrow \infty} \frac{\Delta(x)}{x^{1/4}} = \infty \quad (6)$$

and the complementary result of Ingham or Landau

$$\liminf_{x \rightarrow \infty} \frac{\Delta(x)}{x^{1/4}} = -\infty \quad (7)$$

From analytic number theory point of view, Jacobi (see [4,12]) give a simple evaluation of the function $r_2(n)$. He defined the theta function (see [4,8]): $\vartheta(q) = \sum_{n \in \mathbf{Z}} q^{n^2}$, for $|q| < 1$, then write

$$\vartheta^2(q) = \sum_{(n,m) \in \mathbf{Z} \times \mathbf{Z}} q^{n^2+m^2} = \sum_{n=0}^{\infty} r_2(n) q^n$$

But as proved also by Jacobi (see details in [4]) holds the following theorem

Theorem 1.(Jacobi)

If $q = e^{-\pi\sqrt{r}}$, $r > 0$, then

$$\vartheta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \sqrt{\frac{2K}{\pi}} \quad (8)$$

Hence from the equality

$$\begin{aligned} \frac{2K}{\pi} &= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k_r^2\right) = 1 + 4 \sum_{m=1}^{\infty} \frac{q^m}{1+q^{2m}} = 1 + 4 \sum_{m=1}^{\infty} q^m \sum_{l=0}^{\infty} (-1)^l q^{2ml} = \\ &= 1 + 4 \sum_{m=1}^{\infty} \sum_{l=0}^{\infty} (-1)^l q^{(2l+1)m} \end{aligned} \quad (9)$$

(the function k_r being used here is called elliptic singular modulus and is defined in terms of Weber's functions (see [4,8]) as $k_r^2 = 16q \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^8$) writing

the double series (9) under one sum using divisors sums he obtained the next result

Theorem 2.(Jacobi)

For $n = 1, 2, \dots$ we have

$$r_2(n) = 4 \sum_{d-\text{odd}, d|n} (-1)^{\frac{d-1}{2}} \quad (10)$$

and $r(0) = 1$.

However, here we are interested only in the Gauss circle problem (relation (3)) point of view and not the Dirichlet divisor problem. The Jacobi formula (of Theorem 2) as it stands is very comfortable to use it directly. Although it is very important an estimation of $r_2(n)$ (see Section-2-Lemma 2 below) and finding the desired error term of big- O of relation (3), which as we show is

$$\sum_{n \leq x} r_2(n) = \pi x + o\left(x^{1/4} f(x)\right), \quad x \rightarrow \infty \quad (10.1)$$

as long as $\lim_{x \rightarrow \infty} f(x) = +\infty$.

2 Asymptotic Expansion of $\sum_{n \leq x} r_2(n)$

In this section we give an asymptotic formula for the mean value of $r_2(n)$ using the next formula which is due to Hardy and Voronoi near 1904, (see [9,7])

$$\sum_{n \leq x} r_2(n) = \pi x + x^{1/2} \sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n}} J_1(2\pi\sqrt{nx}) \quad (11)$$

where $J_\nu(z)$ is the general Bessel function of the first kind and ν -th order and given by

$$J_\nu(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \nu + 1)} \left(\frac{z}{2}\right)^{\nu+2k}, \quad 0 \leq |z| < \infty, \quad \nu \in \mathbf{C} \quad (12)$$

If $a, b \in \mathbf{R}$, then we define

$$M_s(a, b) = \sum_{k-\text{odd}, k=1}^{\infty} (-1)^{\frac{k+1}{2}} \frac{\cos(a + b\sqrt{k})}{k^s} \quad (13)$$

$$N_s(a, b) = \sum_{k-\text{odd}, k=1}^{\infty} (-1)^{\frac{k+1}{2}} \frac{\sin(a + b\sqrt{k})}{k^s} \quad (14)$$

and

$$P_s(a, b) = \sum_{n=1}^{\infty} \frac{M_s(a, b\sqrt{n})}{n^s}, \quad Q_s(a, b) = \sum_{n=1}^{\infty} \frac{N_s(a, b\sqrt{n})}{n^s}. \quad (15)$$

Next we prove

Theorem 3.

$$\begin{aligned} R(x) = \sum_{n \leq x} r_2(n) - x\pi &= \frac{x^{1/4}}{\pi} P_{3/4}\left(\frac{\pi}{4}, 2\pi\sqrt{x}\right) + \sum_{s=1}^N \frac{(-1)^s c_1(2s) P_{s+3/4}\left(\frac{\pi}{4}, 2\pi\sqrt{x}\right)}{2^{4s} \pi^{2s+1} x^{s-1/4}} - \\ &- \sum_{s=0}^N \frac{(-1)^s c_1(2s+1) Q_{s+5/4}\left(\frac{\pi}{4}, 2\pi\sqrt{x}\right)}{2^{4s+2} \pi^{2s+2} x^{s+1/4}} + O\left(c_1(2N) 4^{-N} x^{-N-1/2}\right) \end{aligned} \quad (16)$$

where $c_1(m) = (-1)^m \frac{(-\frac{1}{2})_m (\frac{3}{2})_m}{m!}$.

Proof.

From (11) and Theorem 2 we have

$$\begin{aligned} \sqrt{x} \sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n}} J_1(2\pi\sqrt{nx}) &= \sqrt{x} \lim_{N_1 \rightarrow \infty} \sum_{n=1}^{N_1} \frac{r_2(n)}{\sqrt{n}} J_1(2\pi\sqrt{nx}) = \\ &= \sqrt{x} \lim_{N_1 \rightarrow \infty} \sum_{n=1}^{N_1} \left(\sum_{d \text{ odd}, d|n} (-1)^{\frac{d-1}{2}} \right) \frac{1}{\sqrt{n}} J_1(2\pi\sqrt{nx}) = \\ &= \lim_{N_1 \rightarrow \infty} \sqrt{x} \sum_{n(2m-1) \leq N_1} \frac{(-1)^m}{\sqrt{n(2m-1)}} J_1(2\pi\sqrt{n(2m-1)x}) = \\ &= \sqrt{x} \lim_{N_1 \rightarrow \infty} \sum_{p \text{ odd}, np \leq N_1} \frac{(-1)^{\frac{p-1}{2}}}{\sqrt{np}} J_1(2\pi\sqrt{np}x) \end{aligned} \quad (17)$$

The function $J_1(x)$ has the following asymptotic expansion as $x \rightarrow \infty$

$$\begin{aligned} J_1(x) &= \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{3\pi}{4}\right) \sum_{n=0}^{\infty} \frac{(-1)^n c_1(2n)}{(2x)^{2n}} - \right. \\ &\quad \left. - \sin\left(x - \frac{3\pi}{4}\right) \sum_{n=0}^{\infty} \frac{(-1)^n c_1(2n+1)}{(2x)^{2n+1}} \right] \end{aligned} \quad (18)$$

The error due to stopping the summation at any term is the order of magnitude of that term multiplied by $1/x$. Hence, using (18) in (17) we get (16).

Setting $N = 1$ in (16), we get

$$\begin{aligned}
R(x) = & - \lim_{N_1 \rightarrow \infty} \sum_{\substack{np \leq N_1 \\ p - \text{odd}}} \left[\frac{105(-1)^{\frac{p+1}{2}} \sin\left(2\pi\sqrt{np}x + \frac{\pi}{4}\right)}{4096\pi^3(np)^{9/4}x^{5/4}} - \right. \\
& - \frac{15(-1)^{\frac{p+1}{2}} \cos\left(2\pi\sqrt{np}x + \frac{\pi}{4}\right)}{256\pi^2(np)^{7/4}x^{3/4}} + \frac{3(-1)^{\frac{p+1}{2}} \sin\left(2\pi\sqrt{np}x + \frac{\pi}{4}\right)}{8\pi(np)^{5/4}\sqrt[4]{x}} \\
& \left. - \frac{2(-1)^{\frac{p+1}{2}} \sqrt[4]{x} \cos\left(2\pi\sqrt{np}x + \frac{\pi}{4}\right)}{(np)^{3/4}} \right] + O\left(x^{-3/4}\right)
\end{aligned}$$

where $p = 2l + 1$.

Lemma 1.

The Gauss circle problem reduces finding the rate of convergence of $R(x) = \frac{1}{x^{1/4}} \left(\sum_{n \leq x} r_2(n) - \pi x \right)$, which is equivalent to that of

$$\begin{aligned}
S(x) &= \lim_{N_1 \rightarrow \infty} \sum_{n(2l-1) \leq N_1} \frac{(-1)^{l-1} \cos\left(2\pi\sqrt{n(2l-1)}x + \frac{\pi}{4}\right)}{(n(2l-1))^{3/4}} = \\
\lim_{N_1 \rightarrow \infty} \sum_{n=1}^{N_1} \frac{r_2(n) \cos\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right)}{n^{3/4}} &= \sum_{n=1}^{\infty} \frac{r_2(n) \cos\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right)}{n^{3/4}} \quad (19)
\end{aligned}$$

If we manage to show that uniformly

$$S_M(x) = \sum_{n=1}^M \frac{r_2(n) \cos\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right)}{n^{3/4}} < \infty, \quad M \rightarrow \infty, \quad (20)$$

i.e. $S_M(x)$ is uniformly convergent to $S(x)$, then the problem is solved.

Set

$$D_M(x) := \sum_{n=1}^M \frac{\cos\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right)}{n^{3/4-\delta}}, \quad 0 < \delta < \frac{1}{4}, \quad x > 0. \quad (21)$$

We first prove that $r_2(n) = o(n^\epsilon)$, for all $\epsilon > 0$. Then proving $D_M(x)$ is uniformly bounded and uniformly convergent, we take the $\lim_{x \rightarrow \infty} \frac{S(x)}{f(x)} = 0$, for every f such that $\lim_{x \rightarrow \infty} f(x) = +\infty$.

Lemma 2.

For every $\epsilon > 0$ we have

$$r_2(n) = o(n^\epsilon), \quad n \rightarrow \infty \quad (22)$$

Proof.

Let $a = 0, 1, 2, 3$ and $d_a(n) = \sum_{d|n, d \equiv a(4)} 1$. From the known asymptotic formula (see [3] exercises in chapter 14):

$$\sum_{d|n} d^\nu = o(n^{\nu+\epsilon}), \forall \epsilon > 0, n \rightarrow \infty, \quad (23)$$

where $\nu \geq 0$ and

$$r_2(n) = 4(d_1(n) - d_3(n)), \quad (24)$$

we get

$$r_2(n) \leq 4|d_1(n)| + 4|d_3(n)| \leq 8d_0(n) = o(n^\epsilon).$$

which gives the desired result.

Also the Euler-Maclaurin formula for a function F having 4 continuous derivatives in the interval (a, M) states (see [1]):

$$\begin{aligned} \sum_{k=a}^M F(k) &= \int_a^{M+1} F(t) dt + \frac{1}{2} (F(M+a) + F(a)) + \\ &+ \frac{1}{12} (F'(M+a) - F'(a)) - \frac{1}{120} \sum_{k=0}^{M-1} F^{(4)}(a+k+\xi) \end{aligned} \quad (25)$$

with $0 < \xi < 1$.

If we set

$$F(t) = \frac{\cos(2\pi\sqrt{tx} + \frac{\pi}{4})}{t^{3/4-\delta}}, \quad 0 < \delta < \frac{1}{4}$$

then easy

$$\lim_{M \rightarrow \infty} \left| \int_a^{M+1} \frac{\cos(2\pi\sqrt{tx} + \frac{\pi}{4})}{t^{3/4-\delta}} dt \right| < \infty \quad (25.1)$$

Also

$$F'(t) = \left(\delta - \frac{3}{4} \right) \frac{\cos(2\pi\sqrt{tx} + \frac{\pi}{4})}{t^{7/4-\delta}} - \pi\sqrt{x} \frac{\sin(2\pi\sqrt{tx} + \frac{\pi}{4})}{t^{5/4-\delta}}$$

and

$$F^{(4)}(t) = O(t^{-A}), \quad A > 1$$

Hence the Euler-Maclaurin summation formula assures us that we can state the following

Lemma 3.

For every $0 < \delta < \frac{1}{4}$, the $\lim_{M \rightarrow \infty} D_M(x)$ exists and is 'say' $D(x)$.

Lemma 4. (see [14] pg.145)

Suppose that $\lambda_1, \lambda_2, \dots$ is a nondecreasing sequence of real numbers with limit

infinity, and c_1, c_2, \dots is an arbitrary sequence of real or complex numbers, and that $f(x)$ has a continuous derivative for $x \geq \lambda_1$. Put

$$C(x) = \sum_{\lambda_n \leq x} c_n \quad (26)$$

where the summation is over all n for which $\lambda_n \leq x$. Then for $x \geq \lambda_1$,

$$\sum_{\lambda_n \leq x} c_n f(\lambda_n) = C(x)f(x) - \int_{\lambda_1}^x C(t)f'(t)dt \quad (27)$$

Lemma 5.

For every $0 < \delta < \frac{1}{4}$, the sums (21) are uniformly convergent to a bounded function of x , as $M \rightarrow \infty$.

Proof.

Let the function

$$f(t) = \frac{\cos(2\pi t\sqrt{a} + \frac{\pi}{4})}{t^{3/2}} \quad (28)$$

It's first derivative is

$$f'(t) = -\frac{2\pi\sqrt{a}\sin(2\pi\sqrt{a}t + \frac{\pi}{4})}{t^{3/2}} - \frac{3\cos(2\pi\sqrt{a}t + \frac{\pi}{4})}{2t^{5/2}} \quad (29)$$

Setting $y = \sqrt{M}$ we have

$$\sum_{\sqrt{n} \leq y} 1 = y^2 = M \quad (30)$$

Also from Lemma 4 with $y = \sqrt{M}$ we have

$$\begin{aligned} \sum_{\sqrt{n} \leq y} f(\sqrt{n}) &= \left(\sum_{\sqrt{n} \leq y} 1 \right) f(y) - \int_1^y t^2 f'(t) dt = \\ &= \sum_{n \leq M} f(\sqrt{n}) = Mf(\sqrt{M}) - \int_1^{\sqrt{M}} t^2 f'(t) dt \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n \leq M} f(\sqrt{n}) &= \sum_{n \leq M} \frac{\cos(2\pi\sqrt{n}a + \frac{\pi}{4})}{n^{3/4}} = Mf(\sqrt{M}) - \int_1^{\sqrt{M}} t^2 f'(t) dt = \\ &= \frac{1}{\sqrt{2}\sqrt[4]{a}} \left(-2F_C(2\sqrt[4]{a}) + 2F_C(2\sqrt[4]{aM}) + 2F_S(2\sqrt[4]{a}) - 2F_S(2\sqrt[4]{aM}) \right) + \\ &\quad + \frac{1}{\sqrt{2}} (\cos(2\pi\sqrt{a}) - \sin(2\pi\sqrt{a})) \end{aligned} \quad (31)$$

where $F_C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right) dt$ and $F_S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right) dt$ are the Fresnel- C, S functions.

But function (31) is absolutely bounded when $M = 1, 2, \dots$ and $a > 0$ by some constant (we mean $\sqrt{2}$).

Note.

$$\lim_{x \rightarrow \infty} F_C(x) = \lim_{x \rightarrow \infty} F_S(x) = \frac{1}{2} \quad (32)$$

Hence for the sum

$$G(h, x, M) := \sum_{n=1}^M \frac{\cos\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right)}{n^{3/4-h}} \quad (33)$$

it holds that $G(0, x, M)$ is bounded when $M \in \mathbf{N}$ and $x > 0$. Using the mean-value theorem there exists $0 < \xi < \delta$ such that

$$\left| \frac{G(\delta, x, M) - G(0, x, M)}{\delta} \right| = |\partial_h G(h, x, M)|_{h=\xi}$$

Also if we consider the next function

$$f_1(t) = \frac{\cos\left(2\pi t\sqrt{a} + \frac{\pi}{4}\right)}{t^{1+\epsilon}}, \quad \epsilon > 0. \quad (34)$$

We can show as above that (see also Section 3 below)

$$\sum_{n \leq M} f_1(\sqrt{n}) \quad (35)$$

is uniformly bounded and convergent. Hence

$$\begin{aligned} W &= |G(\delta, x, M)| \leq \delta |\partial_h G(h, x, M)|_{h=\xi} + |G(0, x, M)| = \\ &= \delta \left| \sum_{n=1}^M \frac{\cos\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right)}{n^{3/4-\xi}} \log(n) \right| + |G(0, x, M)|. \end{aligned}$$

Assuming that $\xi = \delta - \epsilon > 0$, we have

$$\begin{aligned} W &\leq \delta \left| \sum_{n=1}^M \frac{\cos\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right)}{n^{3/4}} \frac{\log(n)}{n^{\epsilon-\delta}} \right| + |G(0, x, M)| = \\ &O\left(\delta \left| \sum_{n=1}^M \frac{\cos\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right)}{n^{1/2+\epsilon}} \frac{\log(n)}{n^{1/4-\epsilon-\delta+\epsilon}} \right| \right) + |G(0, x, M)| \\ &= O\left(\delta \left| \sum_{n=1}^M \frac{\cos\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right)}{n^{1/2+\epsilon}} \frac{\log(n)}{n^{1/4-\delta}} \right| \right) + |G(0, x, M)| < \infty \quad (36) \end{aligned}$$

uniformly in M and x , when $0 < \delta < \frac{1}{4}$.

Main Theorem.

For every $0 < \delta < \frac{1}{4}$, the series

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{3/4-\delta}} \quad (37)$$

are convergent. Further the partial sums

$$D_M(x) = \sum_{n=1}^M \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{3/4-\delta}} \quad (38)$$

are uniformly bounded and for every function $f(x)$ such that $\lim_{x \rightarrow \infty} f(x) = +\infty$ we have

$$\sum_{n \leq x} r_2(n) = x\pi + o\left(x^{1/4}f(x)\right), \text{ as } x \rightarrow \infty \quad (39)$$

Proof.

The partial sums $D_M(x)$ are uniformly bounded for every $x > 1$, $M \gg 1$, when $0 < \delta < \frac{1}{4}$.

Using [13] Theorem 2 pg. 337, the series in (19) is uniformly convergent, since we can write

$$S(x) = \sum_{n=1}^{\infty} \frac{r_2(n)}{n^{\delta}} \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{3/4-\delta}}. \quad (40)$$

The sequence $\frac{r_2(n)}{n^{\delta}}$, $\delta > 0$ is uniformly convergent to 0 and the remaining term has sum $D(x)$, uniformly convergent and bounded. Hence $S(x)$ is uniformly convergent and for every function $f(x)$ with $\lim_{x \rightarrow \infty} f(x) = +\infty$, we have

$$\lim_{x \rightarrow \infty} \frac{S(x)}{f(x)} = \lim_{x \rightarrow \infty} \sum_{n=1}^{\infty} \frac{r_2(n)}{n^{\delta}} \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{3/4-\delta}f(x)} = 0$$

Note.

An example of such function $f(x)$ is $\log_{\nu}(x) = \underbrace{\log(\log(\dots(\log(x)\dots)))}_{\nu\text{-times}}$, for fixed

large ν or "say" x^{ϵ} , with $\epsilon > 0$, etc.

3 Notes and Remarks

Actually there are more that one can say. For example in the case of the sum (35):

$$D(\epsilon, x, M) := \sum_{n \leq M} \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{1/2+\epsilon/2}}, \text{ where } \epsilon > 0 \quad (41)$$

An evaluation using Lemma 4 in view of the square root method of the proof of Lemma 5, shows that

$$D(\epsilon, x, y^2) = -\frac{(1+i)y^{1-\epsilon}E_{\epsilon}(-2i\pi\sqrt{xy})}{\sqrt{2}} - \frac{(1-i)y^{1-\epsilon}E_{\epsilon}(2i\pi\sqrt{xy})}{\sqrt{2}} +$$

$$\begin{aligned}
& + \frac{(1+i)E_\epsilon(-2i\pi\sqrt{x})}{\sqrt{2}} + \frac{(1-i)E_\epsilon(2i\pi\sqrt{x})}{\sqrt{2}} + \frac{y^{1-\epsilon}\sin(2\pi\sqrt{x}y)}{\sqrt{2}} + \\
& + y^{1-\epsilon}\cos\left(2\pi\sqrt{x}y + \frac{\pi}{4}\right) - \frac{y^{1-\epsilon}\cos(2\pi\sqrt{x}y)}{\sqrt{2}} - \frac{\sin(2\pi\sqrt{x})}{\sqrt{2}} + \frac{\cos(2\pi\sqrt{x})}{\sqrt{2}} \quad (42)
\end{aligned}$$

where $M = y^2$ and

$$E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n} dt \quad (43)$$

is the exponential integral.

Also when $x \geq 1$ and $\epsilon > 0$ we have

$$\begin{aligned}
f(\epsilon, x) &:= \lim_{y \rightarrow +\infty} D(\epsilon, x, y^2) = \\
&= \frac{(1+i)E_\epsilon(-2i\pi\sqrt{x})}{\sqrt{2}} + \frac{(1-i)E_\epsilon(2i\pi\sqrt{x})}{\sqrt{2}} - \frac{\sin(2\pi\sqrt{x})}{\sqrt{2}} + \frac{\cos(2\pi\sqrt{x})}{\sqrt{2}} \quad (44)
\end{aligned}$$

which is a bounded function for all $x \geq 1$. Hence we can write

$$\sum_{n=1}^\infty \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{n^{1/2+\epsilon/2}} = f(\epsilon, x), \quad \epsilon > 0 \quad (45)$$

The case $\epsilon = 0$ lead us to $D(0, x, y^2)$ bounded but not convergent, when $y \rightarrow \infty$. Hence

$$\sum_{n=1}^M \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{\sqrt{n}} \quad (46)$$

is bounded when $M \rightarrow \infty$, but is not convergent, (uniformly bounded). That is not such $\sum_{n=1}^M \{\sin, \cos\}(nx)$. For example with $x = 2$, we get

$$2\pi \sum_{n=1}^M \frac{\cos(2\pi\sqrt{2n} + \frac{\pi}{4})}{\sqrt{n}} = \sin(2\sqrt{2M}\pi) + \cos(2\sqrt{2M}\pi) + C \quad (47)$$

with

$$C = -(1 + \sqrt{2}\pi) \sin(2\sqrt{2}\pi) + (\sqrt{2}\pi - 1) \cos(2\sqrt{2}\pi) \quad (48)$$

and more generally

$$\begin{aligned}
& \sum_{n=1}^M \frac{\cos(2\pi\sqrt{nx} + \frac{\pi}{4})}{\sqrt{n}} = \\
& = \frac{\sin(2\pi\sqrt{xM} + \frac{\pi}{4})}{\pi\sqrt{x}} - \frac{\sin(2\pi\sqrt{x} + \frac{\pi}{4})}{\pi\sqrt{x}} + \cos\left(2\pi\sqrt{x} + \frac{\pi}{4}\right) \quad (49)
\end{aligned}$$

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